63-3-2

THE OCEANOGRAPHIC INSTITUTE
FLORIDA STATE UNIVERSITY
TALLAHASSEE, FLORIDA

EDGE WAVES OVER A SLOPING BEACH IN A ROTATING TWO-LAYERED SYSTEM

Hayato Iida
Takashi Ichiye

Technical Report No. 4

Prepared for Office of Naval
Research Contract No. Nonr-988(11)

April 1963

PROJECT NR 083-16612-62

(Geophysics Branch ONR)

Principal Investigator
Takashi Ichiye

EDGE WAVES OVER A SLOPING BEACH IN A ROTATING TWO-LAYERED SYSTEM

Hayato Iida
Takashi Ichiye
Oceanographic Institute
Florida State University

1. Introduction:

Stokes (1846) showed that a sloping beach can act as a wave guide which allows gravity waves to travel along shore with an amplitude decreasing exponentially with a distance from the shore. Since then, several authors including Ursell (1952), Greenspan (1956) and Munk and others (1956) discussed long or short gravity waves on a sloping beach of a homogeneous water either from a mathematical point of view or from application to geophysical problems.

Kelvin (1879) treated a problem of long waves propagating along a straight coast in a rotating sea and found that the amplitude decreases exponentially from the coast to the left hand side of the direction of propagation in the northern hemisphere. Energy of such waves which are called Kelvin waves, therefore, is seemingly trapped by the coast.

The edge waves over a sloping beach in a rotating sea of homogeneous water was treated by Reid (1958) and Kajiura (1958). They showed that in this system there are two inertio-gravitational waves travelling in opposite directions and one quasi-geostrophic wave.

However, density stratification of the water as well as a beach slope becomes important for problems concerning response of the marginal sea to atmospheric disturbances and for problems concerning internal waves near the coast. In order to represent such stratification in a way which is feasible to mathematical treatment and yet, keeps essential dynamical feature of the system, a two-layer model of the thermocline, consisting of two fluids of different densities is adopted. Deviation of modes of wave motion in a two-layer model from those in a continuous density was discussed by Eckert (1960) in a more or less general way.

Essential features of motion near the coast with a length scale of the width of a continental shelf and with time scales longer than a few hours can be represented by a mathematical model for waves in a two-layered, rotating sea bounded by a straight coast and with a variable depth. There are two types of bottom topography which renders mathematical analysis feasible and yet simulates an area close to the continent.

One type is a continental shelf with a uniform or variable depth connecting with the open sea with a uniform depth as treated by Ichiye (1963). The other type of simple bottom topography is a depth increasing linearly from the coast. In a two-layered sea, there must be longshore geostrophic currents due to pressure gradients if the interface as well as the surface intersects the bottom at the coast.

This study has treated such a model mathematically. In order to simplify the analysis, it is assumed that geostrophic currents in the upper and lower layers have uniform velocities owing to the surface elevation increasing linearly from the coast. Although such an infinite uprise of the sea level seems to be unrealistic, the waves considered have amplitudes significant only near the coast and the simplified model may be justified if interpretation of mathematical results is proper. Slopes of the surface and the interface are assumed so small that the approximation of hydrostatic pressure is valid as shown by Eckart (1951).

2. Fundamental Equations:

The x- and y- axes are taken on the level surface perpendicular and parallel to the coast-line, respectively, and the z-axis is taken vertically downwards, as shown in Fig. 1.

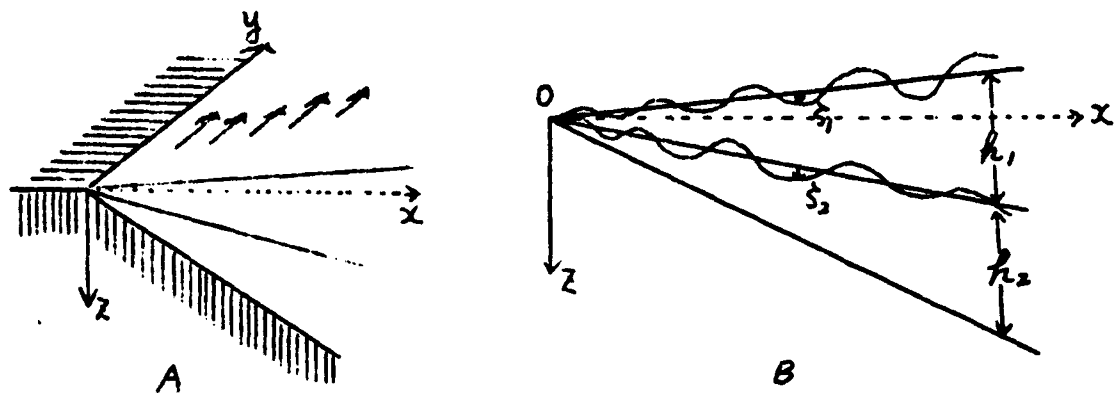


Fig. 1 Schemetic diagrams of coordinate system (A) and a vertical section (B).

It is assumed that densities of water in upper and lower layers are uniform and their difference is small. The basic currents are geostrophic and their velocities are constant in each layer. The waves are considered to be perturbations to such basic currents. The non-linear inertia terms are neglected with assumptions that the vertical velocity and amplitudes of the waves are very small. The vertically integrated, linearized equations of motion without disturbing forces and equations of continuity are respectively as follows:

$$\frac{\partial \mathbb{M}_j}{\partial t} + V_j \frac{\partial \mathbb{M}_j}{\partial y} + \begin{Bmatrix} -f N_j \\ +f M_j \end{Bmatrix} = g h_j \nabla \eta_j \qquad (1)$$

and

$$\nabla \cdot \mathbb{M}_j = -\frac{\partial \phi_j}{\partial t}, \quad \mathbb{M}_j = \{M_j, N_j\}. \qquad (2)$$

where

$$\eta_j = \begin{cases} \rho_1 \zeta_1 & \text{for } j = 1 \\ \rho_1 \zeta_1 + \Delta\rho \zeta_2 & \quad\;\; = 2 \end{cases}$$

and

$$\phi_j = \rho_1(\zeta_2 - \zeta_1) \text{ or } = \rho_2(-\zeta_2) \quad \text{for } j = 1 \text{ or } = 2.$$

In these equations suffices 1 and 2 correspond to the quantities in the upper and lower layer, respectively; M_j, N_j are horizontal mass transports respectively along the x- and y- axes, V_j, the basic current assumed to be geostrophic, ρ_j the density, h_j ($=s_j x$) the thickness of the layer; ζ_1 and ζ_2 are respectively the depression of the surface and the interface from the equilibrium levels, $\Delta\rho$ ($= \rho_2 - \rho_1$) density difference, f the Coriolis parameter, g the gravitational acceleration, and ∇ horizontal gradient operator.

The assumption of geostrophic balance for the basic current leads to

$$V_2 = V_1 - \frac{\Delta}{\rho_2}\frac{g}{f}\frac{dh_1}{dx} . \tag{3}$$

Assuming that horizontal mass transports and displacements of the surface and the interface are proportional to $\exp\{i(ky-\sigma t)\}$ the equations of motions (1) becomes a system of linear equations for M_j and N_j with constant coefficients. The solutions of M_j and N_j in terms of η_j are given by:

$$M_j = i\frac{gh_j}{f^2-\sigma_j^2}\left(-\sigma_j\frac{d\eta_j}{dx} + fk\eta_j\right) \tag{4}$$

and

$$N_j = \frac{gh_j}{f^2-\sigma_j^2}\left(-f\frac{d\eta_j}{dx} + \sigma_j k\eta_j\right) , \tag{5}$$

where $i = \sqrt{-1}$ and $\sigma_j = \sigma - kV_j$.

Substituting M_j and N_j into the equation of continuity (2), we can get equations for ζ_1 and ζ_2 :

$$\left\{D - \left(\frac{fk}{\sigma_1} + \frac{\sigma_1\Delta_1}{s_1 g\sigma_1}\right)\right\}\zeta_1 + \frac{\sigma_1\Delta_1}{s_1 g\sigma_1}\zeta_2 = 0 \tag{6}$$

and $$\left\{D - \left(\frac{fk}{\sigma_2} + \frac{\rho_2}{\Delta\rho}\frac{\sigma_2\Delta_2}{S_2 g \sigma_2}\right)\right\}\zeta_2 + \frac{\rho_1}{\Delta\rho}\left(D - \frac{fk}{\sigma_2}\right)\zeta_1 = 0 \tag{7}$$

where $\Delta_j = f^2 - \sigma_j^2$, $S_j = dh_j/dx$

and $$D = \left[\frac{d}{dx}x\frac{d}{dx} - k^2 x\right].$$

Elimination of ζ_2 from equations (6) and (7) yields the equation for ζ_1:

$$(D^2 - aD + b)\zeta_1 = 0, \tag{8}$$

where $$a = \frac{fk}{\sigma_1} + \frac{\sigma_2\Delta_1}{\delta S_1 g \sigma_1} + \frac{fk}{\sigma_2} + \frac{\sigma_1\Delta_2}{\delta S_2 g \sigma_2} \tag{9}$$

$$b = \frac{1}{\sigma_1\sigma_2}\left(f^2k^2 + \frac{\sigma_1\Delta_1}{\delta S_1 g}fk + \frac{\sigma_2\Delta_2}{\delta S_2 g}fk + \frac{\sigma_1\sigma_2\Delta_1\Delta_2}{\delta S_1 S_2 g^2}\right) \tag{10}$$

and $$\delta = \Delta\rho/\rho_2 .$$

The equation (8) can be separated into two independent Laguerre differential equations:

$$(D - K_1)\zeta_1 = 0 \tag{11}$$

or $$(D - K_2)\zeta_1 = 0, \tag{12}$$

where $$\begin{matrix}K_1\\K_2\end{matrix} = \frac{1}{2}\left(a \pm \sqrt{a^2 - 4b}\right). \tag{13}$$

It is seen that $a^2 - 4b \geq 0$ for small δ.

In order that the solutions of equations (11) and (12) may be finite both at the coast and at the infinite distance, the K_j must equal to $-(n + 1)k$, where $\underline{n}$ is an integer. These conditions yield the relation between ζ_1 and ζ_2 and the frequency equation, respectively:

$$\zeta_2 = \frac{s_1 g}{\Delta_1}\left\{(2n+1)k + \frac{fk}{\sigma_1} + \frac{\Delta_1}{s_1 g}\right\}\zeta_1 \tag{14}$$

and
$$\begin{aligned}&(2n+1)^2\frac{\delta s_1 s_2 g^2}{f^4}k^2\omega_1\omega_2 + (2n+1)\left[\frac{\delta s_1 s_2 g^2}{f^3}k^2(\omega_1+\omega_2) + \right.\\ &\left. + \left\{\frac{s_1 g}{f^2}\omega_1\omega_2(1-\omega_2^2) + \frac{s_2 g}{f^2}\omega_2\omega_1(1-\omega_1^2)\right\}\right] + \frac{s_1 s_2 g^2}{f^2}k^2 + \\ &+ \frac{s_1 g}{f^2}k\,\omega_2(1-\omega_2^2) + \frac{s_2 g}{f^2}k\,\omega_1(1-\omega_1^2) + \omega_1\omega_2(1-\omega_1^2)(1-\omega_2^2) = 0\end{aligned} \tag{15a}$$

The frequency equation (15a) can be reduced into

$$\begin{aligned}&\left[\frac{\delta s_1 g}{f^2}k\left\{(2n+1)\omega_1+1\right\} + \omega_1(1-\omega_1^2)\right]\left[\frac{\delta s_2 g}{f^2}k\left\{(2n+1)\omega_2+1\right\} + \omega_2(1-\omega_2^2)\right]\\ &\qquad = (1-\delta)\omega_1\omega_2(1-\omega_1^2)(1-\omega_2^2)\end{aligned} \tag{15b}$$

In these equations following notations are used:

$$\omega = \frac{\sigma}{f}, \quad \omega_i = \frac{\sigma_i}{f} = \omega - \frac{kV_i}{f} = \omega - k'$$

and

$$\frac{\Delta_i}{f^2} = 1 - \frac{\sigma_i}{f^2} = 1 - \omega_i^2 .$$

The equation (15b) can be written in the polynomial about w. The coefficients of different powers of w are shown in the following Table 1, in which the dimensionless quantities, ε, k', and γ are defined by

$$\varepsilon = -\frac{V_2}{V_1}, \quad k' = \frac{V_1 k}{f} \text{ and } \gamma = \frac{s_1+s_2}{s_1}$$

Equation (3) yields $\quad (1+\varepsilon)k' = (V_1 - V_2)kf^{-1} = g\delta(s_1+s_2)kf^{-1}$

Therefore $(1+\varepsilon)k'$ is always positive, no matter in what direction the current in upper and lower layers flow. Values of for different values of V_1 and wave length are shown in Table II, for $f = 10^{-4}$ (sec^{-1}).

Table I Frequency Equations

ω^n	Coefficients
ω^6	1
ω^5	$-3(1-\varepsilon)k'$
ω^4	$3(1-3\varepsilon+\varepsilon^2)k'^2-(2n+1)(1+\varepsilon)\frac{1}{\delta}\gamma k'-2$
ω^3	$-(1-\varepsilon)(1-4\varepsilon+\varepsilon^2)k'^3+(2n+1)(1+\varepsilon)\frac{1}{\delta}\{(3-\gamma)(\gamma-1)+(1-\varepsilon)\}k'^2+$ $+\{3(1-\varepsilon)-(1+\varepsilon)\frac{1}{\delta}\gamma\}k'$
ω^2	$-3(1-3\varepsilon+\varepsilon^2)k'^4+3(2n+1)(1-\varepsilon^2)\frac{1}{\delta}(\gamma-1-3\varepsilon+\varepsilon^2)k'^3+$ $+(1+\varepsilon)\frac{1}{\delta}\{(2n+1)^2(1+\varepsilon)(\gamma-1)+3(\gamma-1-\varepsilon)\}k'^2+(2n+1)(1+\varepsilon)\frac{1}{\delta}\gamma k'+1$
ω^1	$-3\varepsilon^2(1-\varepsilon)k'^5-(2n+1)(1+\varepsilon)\frac{1}{\delta}\{(\gamma-1)(3\varepsilon-1)+\varepsilon^2(\varepsilon-3)\}k'^4-$ $-(1+\varepsilon)\frac{1}{\delta}\{(2n+1)(1-\varepsilon^2)(\gamma-1)+3(\varepsilon^2+\gamma-1)\}k'^3+3(2n+1)(1+\varepsilon)^2(\gamma-1)\frac{1}{\delta}k'^2+$ $+(1+\varepsilon)\frac{1}{\delta}\gamma k'$
ω^0	$-\varepsilon^3k'^6+(2n+1)(1+\varepsilon)\frac{1}{\delta}\varepsilon(1+\varepsilon^2)\gamma k'^5$ $-(2n+1)^2(1+\varepsilon)^2\varepsilon(\gamma-1)\frac{1}{\delta}k'^4-(2n+1)(1+\varepsilon)\frac{1}{\delta}\{(1-\varepsilon^2)(\gamma-1)+\varepsilon\gamma\}k'^3+$ $+\frac{1}{\delta}(1+\varepsilon)\varepsilon\gamma k'^2$

Table II Nondimensional wave number $k'=V_1k/f$ corresponding to a given wave length L (km) and velocity V_1(m/sec), where $f = 10^{-4}sec^{-1}$.

V_1 \ L	10	50	100	500	1000	5000 km
1 m/s	6.283	1.257	0.628	0.126	0.063	0.013
0.8	5.026	1.066	0.503	0.101	0.050	0.010
0.5	3.142	0.628	0.314	0.063	0.031	0.006
0.2	1.257	0.251	0.126	0.025	0.013	0.003
0.1	0.628	0.126	0.063	0.013	0.006	0.001

3. The waves for n = 0

When n = 0, the polynomial (15b) can be factorized as follows

$$
\begin{aligned}
&(\omega-k'+1)(\omega+\varepsilon k'+1)\times \\
&\Big[\{(\omega-k')(\omega-k'+1)-(1+\varepsilon)k'\}\{(\omega+\varepsilon k')(\omega+\varepsilon k'-1)+(1+\varepsilon)(\gamma-1)k'\} \\
&\qquad +(1-\delta)(\omega-k')(\omega+\varepsilon k')(\omega-k'+1)(\omega+\varepsilon k'-1)\Big]
\end{aligned}
\tag{16}
$$

It is seen at once that equation (16) has two roots:

$$\omega = k'-1 \tag{17}$$

$$\omega = -\varepsilon k'-1 \tag{18}$$

The root (17) or (18) makes the denominator $f^2-\sigma_j$ in equations (4) and (5) vanish for j = 1 or 2, respectively. Therefore, equations (4) and (5) yield $\eta_j \sim e^{-kx}$

in which the suffix j equals to 1 or 2, corresponding to the root (17) or (18), respectively.

Substituting η_j into either (6) or (7), it is found that ζ_1 and ζ_2 are proportional to e^{-kx}. The mass transport M_1 or M_2 respectively for the root (17) or (18) can be derived from one of equations of motion (1) and equations of continuity (2). The boundary condition that M_j thus determined should vanish at the coast and be finite at an infinite distance leads to a relation which determines the values of $\underline{k}$.

One such corresponding to the root (17) is given by

$$\delta S_1 S_2 g^2 k'^2 + f^2\omega_2(1-\omega_2)gS_1 k + 2S_2 g f^2\omega_1 k + 2f^4\omega_1\omega_2(1-\omega_2) = 0 \tag{19}$$

Since w_1 = -1 and w_2 is linear with $\underline{k}$, equation (19) is a cubic equation about $\underline{k}$. This equation yields the roots when δ is very small,

$$k \sim \frac{2f^2}{(\varsigma_1+\varsigma_2)g} + O(\delta), \tag{20a}$$

$$k \sim \frac{2f^2}{\delta(3\varsigma_1+\varsigma_2)g} + O(1), \tag{20b}$$

and

$$k \sim \frac{2(3\varsigma_1+\varsigma_2)f^2}{\delta^2(\varsigma_1)^2 g} + O\left(\frac{1}{\delta}\right). \tag{20c}$$

Relation (20a) is equal to $2f^2/g(\varsigma_1+\varsigma_2)$ when $\delta = 0$ and $V_1 = V_2 = 0$, as obtained by Reid (1958).

The waves corresponds to the roots (17) and (18) are degenerate cases of other modes of waves as in case of a uniform density. Since these waves are possible only for discrete wave numbers, they cannot be excited by local atmospheric disturbances of arbitrary dimensions.

Other four roots of $\underline{w}$ in equation (16) are determined from the following biquadratic equation with $\underline{w}$:

$$\begin{aligned} &\omega^4 - 2\{(1-\varepsilon)k'+1\}\omega^3 - \{2\varepsilon k'^2+(1+\varepsilon)\gamma\delta^{-1}k'-3(1-\varepsilon)k'-1\}\omega^2 \\ &+\{-2(\varepsilon-1)k'^3-(\varepsilon^2-4\varepsilon+1)k'^2+2(1+\varepsilon)(\gamma-1-\varepsilon)\delta^{-1}k'^2-(1-\varepsilon)k'+ \\ &+(1+\varepsilon)\gamma\delta^{-1}k'\}\omega + \{\varepsilon^2k'^4+\varepsilon(\varepsilon-1)k'^3-(1+\varepsilon)(\gamma-1+\varepsilon^2)\delta^{-1}k'^3- \\ &-\varepsilon k'^2+(1+\varepsilon)\varepsilon\gamma\delta^{-1}k'^2 = 0 \end{aligned} \tag{22}$$

As seen from Table II, $k' \ll \delta^{-1}$ for the wave lengths longer than several kilometers and reasonable values of V_1 because $\delta \approx 10^{-3}$ In such a range of k' roots of equation (22) can be expressed approximately with algebraic relations of k'.

First, only terms with δ^{-1} are kept for the same powers of k', since $\delta \ll 1$. Further, if $k' \ll 1$, powers of k' higher than the first in the coefficients of $\underline{w}^2$ and $\underline{w}$ can be neglected. The left hand side of equation (22) can be written as

$$\omega(\omega-1)\left[\omega^2-\{1+2(1-\varepsilon)k'\}\omega-(1+\varepsilon)\delta^{-1}\gamma k'\right] \\ -(1+\varepsilon)\varepsilon\gamma\delta^{-1}k'^2+(1+\varepsilon)(\gamma-1)\delta^{-1}k'^3=0 \tag{23}$$

If the last two terms of equation (23) are neglected, the first approximation of four roots of (23) is given by:

$$\omega \approx 0 \text{ and } \omega \approx 1 , \qquad (24a)\ (24b)$$

and

$$\omega \approx \frac{1}{2} \pm \left\{\frac{1}{4}+(1+\varepsilon)\frac{\gamma}{\delta}k'\right\}^{1/2} , \qquad (25a)\ (25b)$$

In relations (25a) and (25b), k' -terms without a factor δ^{-1} are neglected. The following second approximations for (24a) and (24b) are determined from (22) to the order of k'^2 and k', respectively.

$$\omega \sim -\varepsilon k' + \frac{(\gamma-1+\varepsilon^2)}{\gamma}k'^2 \tag{26a}$$

$$\omega \sim 1+\{1+(1-2\gamma^{-1})(1+\varepsilon)\}k' \tag{26b}$$

The term of k'^2 in (26a) becomes important when $\varepsilon=0$. The roots (26a) and (26b) yield respectively following phase velocities:

$$C=\frac{\omega}{k'}V_1 = V_2 + \frac{(\gamma-1+\varepsilon^2)}{\gamma}V_1 k' \tag{27a}$$

$$C = V_1 + \left(\frac{V_1}{k} + \frac{\gamma-2}{\gamma}(1+\varepsilon)V_1\right) \tag{27b}$$

The depression ζ_2 of the interface corresponding to (26a) (26b) are respectively given by

$$\zeta_2 \approx -\frac{1}{\delta}\zeta_1 \tag{28a}$$

$$\zeta_2 \approx -\frac{1}{\delta}\frac{\gamma}{\gamma-2}\zeta_1 \tag{28b}$$

Since equations (28a) and (28b) indicate that displacements of the interface are much larger than and reserve in direction to those of the free surface, the waves corresponding to roots (26a) and (26b) are interpreted as the baroclinic mode. The frequency corresponding to (26a) is very small and the mass transports which are obtained by substituting (26a) into (4) and (5) are quasigeostrophic or geostrophic to the order of magnitude of k' . Since the frequency for the root (26b) almost equals to that of inertial oscillations, waves corresponding to this root may be called inertiogravitational waves. The phase velocity (27a) of quasigeostrophic waves shows that the waves are carried by the current of the lower layer. When $V_2 = 0$, they propagate to the positive y-direction. When $V_2 \neq 0$, they may propagate to the negative y-direction for smaller k' but to the positive y-direction for larger k' in a range of k' satisfying $k' \ll 1$. The value of k' at which such change in propagation direction occurs is given by $\varepsilon\gamma(\gamma-1+\varepsilon^2)^{-1}$ which lies in the range of k' considered if ε is sufficiently small. Inertiogravitational waves are carried by the current of the upper layer, as seen from equation (27a).

Dynamical properties of the waves corresponding to (25) may be understood when we consider a range of k' satisfying $k' \ll \delta(1+\varepsilon)^{-1}\gamma^{-1}$. Under this condition, the roots (25) can be expressed by

$$\omega \simeq -(1+\varepsilon)\gamma\frac{1}{\delta}k' \tag{29a}$$

$$\omega \approx 1+(1+\varepsilon)\gamma\frac{1}{\delta}k' \tag{29b}$$

The displacement of the interface corresponding to (29a) and (29b) is given by:

$$\zeta_2 = \frac{\gamma-1}{\gamma}\zeta_1 \tag{30}$$

which indicates that the waves are of barotropic mode. The same argument as for the waves corresponding to (26a) and (26b) can be applied to the waves of barotropic mode and waves for (29a) or (29b) may be defined as quasi-geostrophic or inertiogravitational waves, respectively.

When $\delta |1+\varepsilon|^{-1}\gamma^{-1} \ll k' \ll \gamma\delta^{-1}\left|\frac{(1+\varepsilon)}{\varepsilon}\right|$, equation (22) can be approximately separated into two quadratic equations:

$$\omega^2 - 2\{1+(1-\varepsilon)k'\}\omega - (1+\varepsilon)\frac{\gamma}{\delta}k' = 0 \qquad (31)$$

$$\gamma\omega^2 - \{2(\gamma-1-\varepsilon)k' + \gamma\}\omega + \{-(\gamma-1+\varepsilon^2)k'^2 + \varepsilon\gamma k'\} = 0 \qquad (32)$$

The roots of (32) represent the waves with high frequencies which become waves of barotropic mode corresponding to the roots (29a) and (29b) as k' decreases to the order less than δ. The waves corresponding to the roots of equation (31) have lower frequencies than those for (32). They become waves of baroclinic mode corresponding to the roots (28a) and (28b) as k' decreases.

The four roots of (22) for a range of k' from 10^{-4} to 10 are numerically computed by taking following constants.

$$\gamma = 5, \quad \varepsilon = 0, \ 0.5, \ 1 \text{ and } \infty, \quad \delta = 10^{-3} \qquad (33)$$

The curves of $\underline{w}$ are plotted against in fig. 2a, b. The values 0 and ∞ of ε corresponds to the cases of no current in the lower and upper layer, respectively. Since k' vanishes for $\varepsilon=\infty$, k' is replaced by $k''(=k\bar{V}_2 f^{-1})$ in fig. 2b. It is seen that general features of four curves of $\underline{w}$ are very similar for different values of ε except those of quasigeostrophic waves of baroclinic mode. Since the prominent terms in most approximate formulas for $\underline{w}$ include $(1+\varepsilon)k'$ as a combination, the curves of $\underline{w}$ are almost the same for negative values of $(1+\varepsilon)$ as for positive values if $|k'|$ is taken as an argument.

In fig. 3 the ratio $|\zeta_2/\zeta_1|$ for four different roots of the biquadratic equation in (16) are plotted against k' for a case of $\varepsilon = 0$. These ratios are almost unchanged in a range of considered. Although explicit formulas of ζ_2 indicating separation of barotropic and baroclinic modes such as (28) and (30) are derived only for a certain range of k', such separation is almost complete in the whole range of k'. There seems to be no important interaction between barotropic and baroclinic modes.

Since the waves of barotropic mode are similar to those treated by Reid (1958), the approximate formulas of the roots of this mode can be compared with his results. From the definition of ε, k', γ and δ, we have

$$(1+\varepsilon)\gamma\delta^{-1}k' = g\frac{(S_1+S_2)}{f^2}k \tag{34}$$

If we put $S = S_1 + S_2$, the roots expressed by (25) become the same as those of equations (35) and (36) in Reid's (1958) paper. (Note that his notation $\underline{w}$ is equivalent to $-\sigma$ or $-f\omega$ in our notation.) As k' or k increases, these roots can be approximately expressed by roots of equation (31), which are equivalent to those of equation (40) in Reid's paper.

Since $(1+\varepsilon)k'$ is positive from relation (34) and $\gamma > 1$ from the definition, the approximate formulas (23), (31) and (33) always yield real values of $\underline{w}$. This situation indicates that the waves for n = 0 are stable for a range of wave numbers satisfying the condition that $0 < k \ll \delta^{-1} f V_1^{-1}$. This range includes waves whose wave lengths range from thousands to several kilometers, for ordinary magnitudes of δ ($\sim 10^{-3}$) and V_1 (~ 1/m/s). Such persistence of stability in the range of wave lengths is different from occurrence of long unstable waves which are predicted in a mathema-

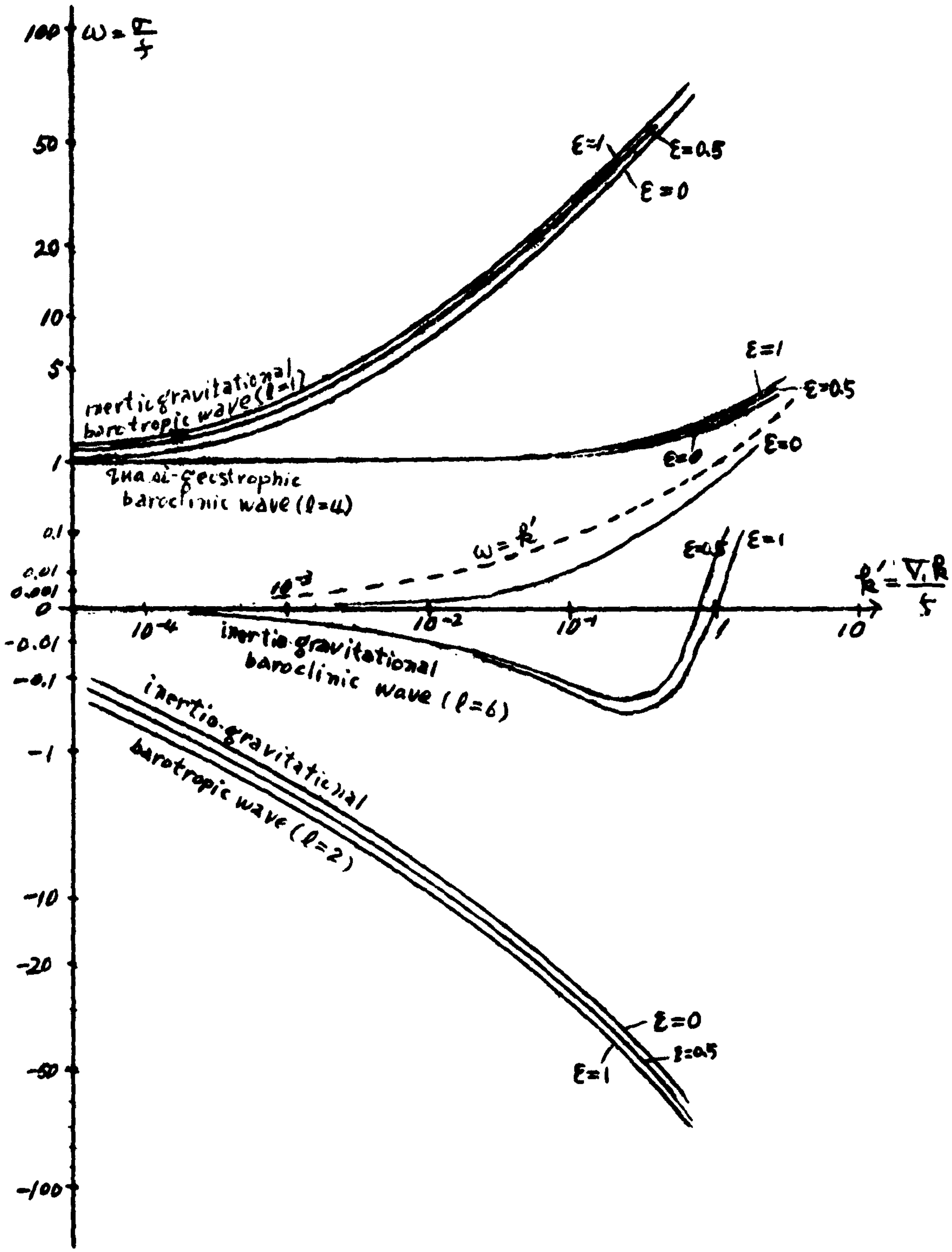


Fig. 2a.--Relative frequencies $\omega_{0\ell} = \sigma_{0\ell}/f$ versus $k' = \frac{\nabla_1 k}{f}$ for ε = 0, 0.5 and 1.

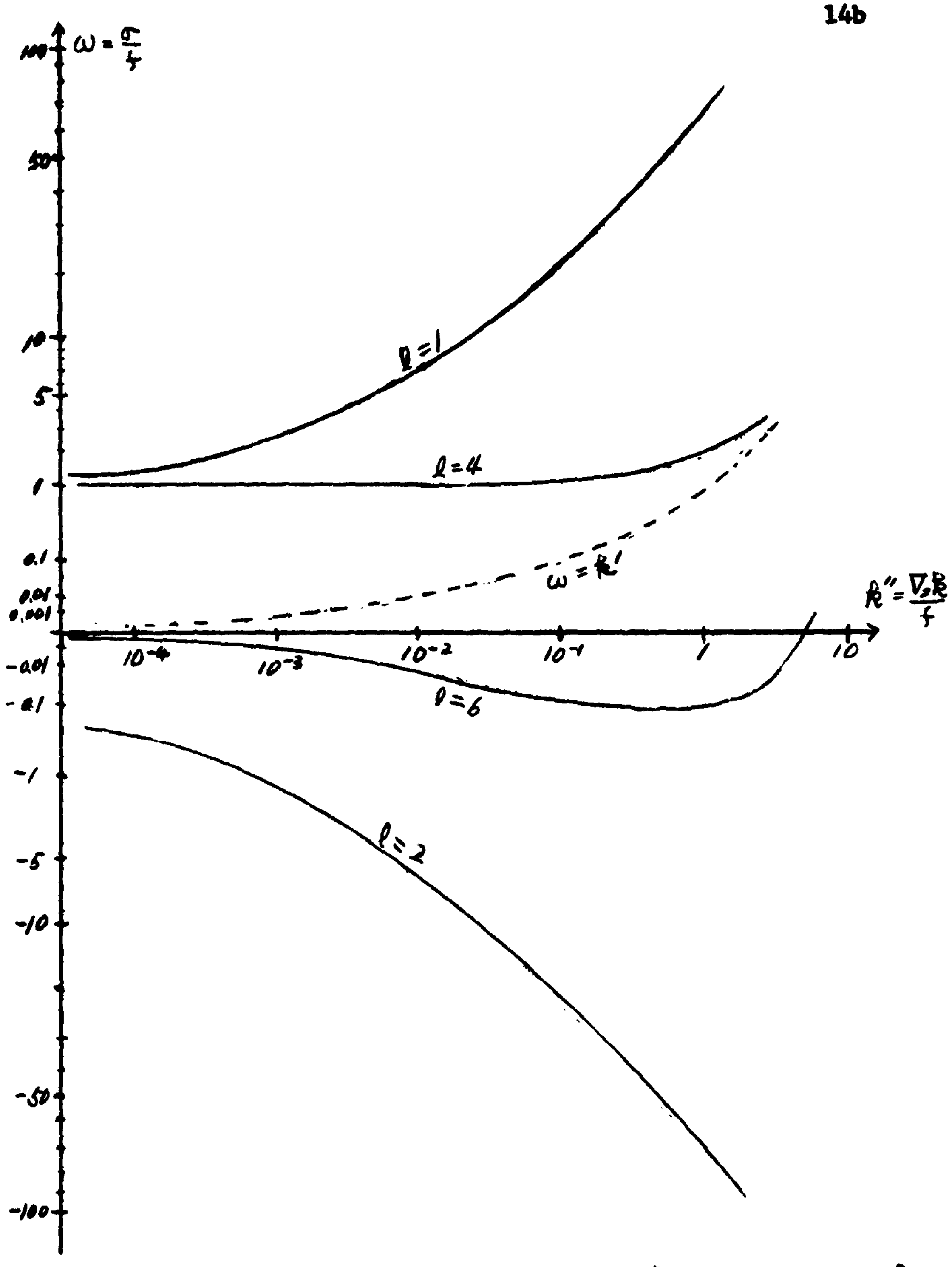


Fig. 2b.--Relative frequencies $\omega_{0\ell} = \sigma_{0\ell}(k'')/f$ versus $k'' = \nabla_2 k/f$ for $\varepsilon = \infty$.

tical model of cyclones as waves on an inclined boundary of two air masses in the atmosphere (Godske and others, 1957). This difference might certainly be due to the kinematic constraints imposed by the bottom configuration, the free surface, and the coast in the present model, though a real reason must be yet to be found in more elaborate study. (See Appendix A)

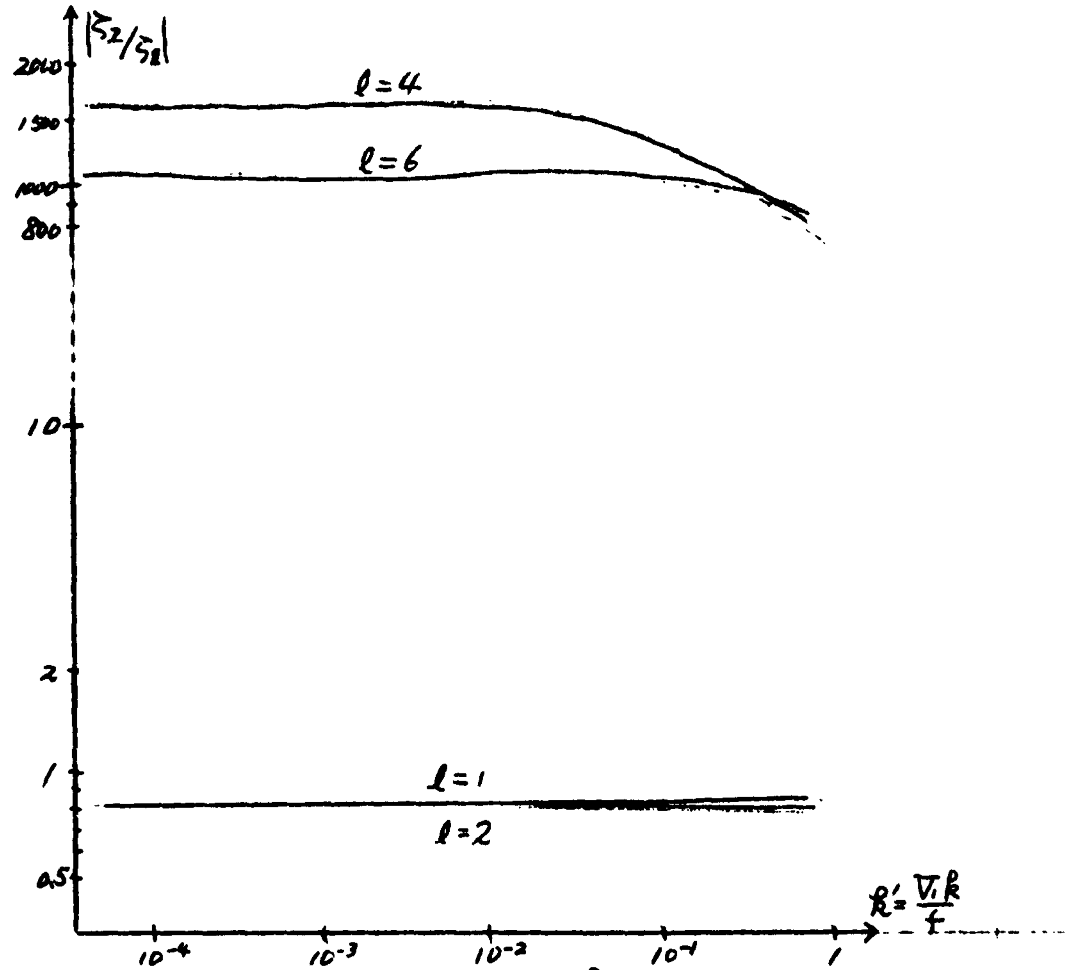


Fig. 3.-- $|\zeta_2/\zeta_1|$ versus $k' = \frac{V_1 k}{f}$ for n = 0 and $\varepsilon = 0$.

4. The waves for $n \geq 1$

When k' is much smaller than $\delta(2n+1)^{-1}$, equation (15b) yields approximate relations for ω. Approximation for three roots in a low frequency range is obtained by equating to zero respectively the linear and quadratic part about ω in equation (15b), in which only the zeroth, first and second powers of k' are retained respectively in the coefficient of the second, first and zeroth power of ω in the polynomial shown in Table 1. These roots are:

$$\omega \sim -\varepsilon k' + (2n+1)\{(1-\varepsilon^2)(\gamma-1)\gamma^{-1}+\varepsilon\}k'^2 \tag{35a}$$

$$\omega \sim \pm 1 \pm \gamma^{-1}\{(2n+1\pm 1)(1+\varepsilon)(\gamma-1)\pm\gamma\}k' \tag{35b}$$

As the roots (26a) and (26) for $n = 0$, these roots represent motion of baroclinic mode which is characterized by the values of ζ_2 proportional to $\delta^{-1}\zeta_1$. Substitution of (35a) and (35b) into (4) and (5) as well as the magnitude of characteristic frequencies confirm that the roots (35a) and (35b) correspond respectively to quasigeostrophic ($\ell = 6$) and inertio-gravitational waves ($\ell = 4$ and 5), where ℓ denotes the mode of waves for different roots of (15b) with $\ell = 1$ to 3 and 4 to 6 corresponding to barotropic and baroclinic mode, respectively.

The three roots of (15b) in a high frequency range is determined from a cubic equation:

$$\omega^3 - 3(1-\varepsilon)k'\omega^2 - \{1+(2n+1)(1+\varepsilon)\delta^{-1}\gamma k'\}\omega - \frac{1}{\delta}(1+\varepsilon)\gamma k' = 0 \tag{36}$$

If the term of ω^2 is neglected, this equation becomes the same as equation (30) in the paper of Reid (1958) by putting $S = S_1 + S_2$.

Therefore, these roots correspond to the motion of barotropic mode. The approximate formulas for three roots of (36) can be written as:

$$\omega \sim -\delta^{-1}(1+\varepsilon)\gamma k' \tag{37a}$$

$$\omega \sim \pm 1 \pm (1+\varepsilon)\gamma \delta^{-1}\{(2n+1)\pm 1\} k' \tag{37b}$$

with the condition $k' \ll \delta$. The same argument as applied to the roots (35a) and (35b) indicates that the roots (37a) and (37b) correspond to quasigeostrophic ($\ell = 3$) and inertiogravitational waves ($\ell = 1$ and 2), respectively.

In a range of satisfying the condition that $\delta(2n+1)^{-1} \ll k' \ll \delta^{-1}(2n+1)$ equation (15b) can be separated into a quadratic and biquadratic equation of ω . The quadratic equation is

$$\omega^2 - (2n+1)(1+\varepsilon)\delta^{-1}\gamma k' \simeq 0 . \tag{38a}$$

If S is substituted for $S_1 + S_2$ and k is used instead of k', this equation becomes:

$$\omega^2 - (2n+1) g S k \simeq 0 \tag{38b}$$

Therefore the roots of (38b) represent the gravitational waves of barotropic (ℓ=1 and 2) similar to Stokes' edge waves.

The biquadratic equation of ω is obtained by equating to zero the sum of all the terms with a factor δ^{-1} in Table 1. If $\delta(2n+1)^{-1} \ll k' \ll (2n+1)^{-1}$, this equation has the following four roots:

$$\omega \sim (2n+1)\{(1-\varepsilon^2)(\gamma-1)\gamma^{-1} - \varepsilon\} k'^2 - \varepsilon k' \tag{39a}$$

$$\omega \sim \pm 1 + O(k') \tag{39b}$$

$$\omega \sim -\frac{1}{2n+1} + O(k') . \tag{39c}$$

The roots given by (39a) and (39b) correspond respectively to (35a) and (35b) and represent quasigeostrophic ($\ell = 6$) and inertiogravitational ($\ell = 4$ and 5) waves of baroclinic mode. The root (39c) represents quasigeostrophic waves of barotropic mode ($\ell = 3$) for $k \gg f^2(2n+1)^{-1}(gS)^{-1}$ in Reid's (1958) paper. As k' approaches to and exceed $(2n+1)^{-1}$, roots of the biquadratic equation are no more adequately expressed by approximate formulas (39a) to (39c). The formulas for such roots cannot be determined as simple algebraic relations of k'.

In figure 4, the curves of six roots of equation (15b) with $n = 1$ are plotted for k'. A range of k' and constants δ and γ are the same as for the curves of figure 2 of $n = 0$. The curves of roots of barotropic mode have features similar to those of Reid's (1958) paper, indicating that there is little interaction between barotropic and baroclinic modes in a range of k' considered. Also, numerical calculation as well as approximate formulas for ω indicate that the roots of equation (15b) are real for $n \geq 1$ and the waves are stable in this range of k'.

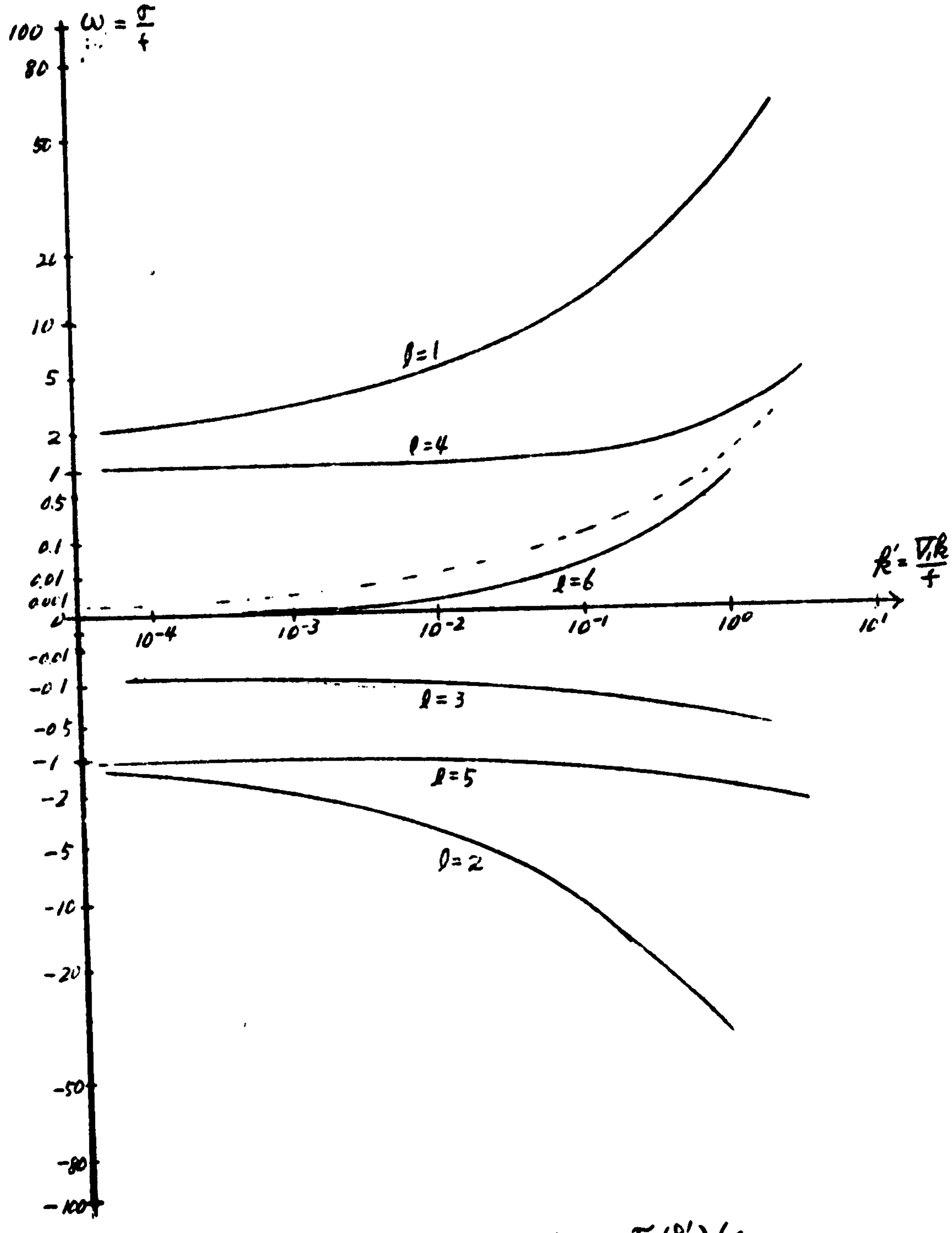


Fig. 4.--Relative frequencies $\omega_{1l} = \sigma_{1l}(k')/f$ versus $k' = \nabla_1 k / f$ for $\varepsilon = 0$.

5. Group velocity.

The group velocity $G_{n\ell}(k)$ of the waves corresponding to the root $\omega_{\ell n}(k')$ of equation (15b) is given by

$$G_{\ell,n}(k') = \frac{d\sigma_{\ell n}}{dk} = \frac{d\omega_{\ell n}}{dk'} V_1 \tag{40}$$

in which $\sigma_{\ell,n}$ is the frequency corresponding to $\omega_{\ell,n}$ and the suffices ℓ (= 1 to 6) and n (=0,1,2...) are the mode and the order of the waves. If $F(\omega, k')$ represents the frequency equation whose coefficients of powers of ω are listed in Table 1, the group velocity (40) is expressed by:

$$G_{\ell,n}(k') = V_1 \left\{ -\left(\frac{\partial F}{\partial k'}\right) \Big/ \left(\frac{\partial F}{\partial \omega}\right) \right\}_{\omega = \omega_{\ell n}} \tag{41}$$

General features of group velocity of waves of different modes may be seen from the slope of the curve of $\omega_{\ell n}(k')$ plotted against k' as well as approximate formulas for $\omega_{\ell n}(k')$. Trends of roots $\omega_{0n}(k')$ of barotropic mode as functions of k' are almost similar to those discussed by Reid (1958) in a range of k' considered in this study. Thus, the group velocities of the barotropic modes have similar features to those of his theory.

In order to discuss the group velocities of baroclinic mode for n = 0, approximate formulas of ω are determined from equation (32) for two ranges of k'. In a range $1 \gg k' \gg \delta$, roots of (32) are given by

$$\omega \approx -\varepsilon k' + \{(1-\gamma^{-1})(1+2\varepsilon) - \varepsilon^2\gamma^{-1}\} k'^2 \tag{42a}$$

$$\omega \approx 1 + 2(1-\gamma^{-1} - \gamma^{-1}\varepsilon) k' \tag{42b}$$

In a range $\delta^{-1} \gg k' \gg 1$, these roots are expressed by:

$$\omega \simeq (1-\gamma^{-1}-\varepsilon\gamma^{-1})k' + \frac{1}{2} \pm \{q_1 k' + q_2\} \qquad (43a)\ (43b)$$

where
$$q_1^2 = \{(1-\gamma^{-1})(2\gamma-2\varepsilon+1)+\varepsilon^2(1-\gamma^{-1})\}\gamma^{-1}$$
$$q_2 = (1-\gamma^{-1}-\varepsilon-\varepsilon\gamma^{-1})(2q_1)^{-1} .$$

The formula with minus or plus sign before the double sign is denoted by (43a) or (43b), respectively. Approximate formulas of ω in the quasigeostrophic waves of baroclinic mode are expressed by (26a) (42a) and (43a) in respective ranges of k'. Those of inertiogravitational wave are given by (26a), (42b) and (43b).

The group velocity as well as the phase velocity for the quasigeostrophic waves is smaller than corresponding velocities of the inertiogravitational waves in a range of k' considered as seen from relations (26), (42) and (43). If ε is finite and of order of magnitudes less than unity, the group velocity of the quasigeostrophic waves is almost the same as the phase velocity which equals nearly to V_2, velocity of the permanent flow in the lower layer in a range $k'' \ll 1$. Therefore, with such a condition these waves are almost non-dispersive. If $\varepsilon \approx 0$, in a range $k' \ll 1$ the group velocity of quasigeostrophic waves are twice the phase velocity which is the order of magnitude of $k' V_1$. In a range $\delta^{-1} \gg k' \gg 1$, group velocity is almost the same as the phase velocity for any values of ε.

The group velocity of the inertiogravity waves equals approximately to $2V_1$ in a range $k' \ll 1$, and to $2\sqrt{2}\,V_1$ in a range $1 \ll k' \ll \delta^{-1}$, if $\gamma \gg 1$ and $\varepsilon \ll 1$. It is much smaller in a range $k' \ll 1$ than the phase velocity which is nearly equal to $V_1/k' (= f k^{-1})$. Therefore, these waves are strongly dispersive in such a range of k'. They become less dispersive in a range $1 \ll k' \ll \delta^{-1}$ as in the quasigeostrophic waves.

Analogous discussions may be applied to the group velocity of the waves of order higher than zeroth order. The group velocity of the second inertiogravity waves corresponding to the root with the lower sign of (35b) and (39b) is negative as the phase velocity is , although the magnitude of the group velocity is much smaller than the phase velocity in a range $k' \ll 1$. In this range of k', therefore, these waves are highly dispersive as the first inertiogravitational waves for roots with the upper sign of (35b) and (39b).

6. Normal modes

Arbitrary waves in this systen can be expressed as a sum of the waves of normal modes of ℓ = 1 to 6 for all order n from 0 to infinity, because orthogonal relationships among normal móde functions are valid as proved in Appendix B. In order to discuss the waves generated by an initial mound of the free surface or by wind stresses or atmospheric pressure, the initial surface elevation or forcing functions of wind stresses or atmospheric pressure must be expressed by a sum of normal mode functions.

The orthongonal conditions for the present system are expressed as follows (see Appendix B for derivation):

$$\int_0^\infty \Big[\frac{1}{\rho_1 \rho_2 g h_1} \{ M_{1,n\ell} M^*_{1,mp} + N_{1,n\ell} N^*_{1,mp} \} + \frac{\rho_1}{\rho_2} \zeta_{1,n\ell} \zeta^*_{1,mp} + \tag{44}$$
$$+ \frac{1}{\rho_2^2 g h_2} \{ M_{2,n\ell} M^*_{2,mp} + N_{2,n\ell} N^*_{2,mp} \} + \frac{\Delta\rho}{\rho_2} \zeta_{2,n\ell} \zeta^*_{2,mp} +$$
$$+ \frac{\rho_1}{\rho_2} (\bar{V}_1 - \bar{V}_2) k^2 \rho_1 g \left(\frac{(2n+1)\sigma_{1,n\ell}\sigma_{1,mp}(\sigma_{1,mp} \pm \sigma_{1,n\ell}) - f(\sigma_{1,mp}^2 + \sigma_{1,mp}\sigma_{1,n\ell} + \sigma_{1,n\ell}^2) + f^3}{\sigma_{1,n\ell}\sigma_{1,mp}(f^2 - \sigma_{1,n\ell}^2)(f^2 - \sigma_{1,mp}^2)} \right) \zeta_{1,n\ell} \zeta_{1,mp} \Big] dx = 0$$

in which $M_{j,n\ell}$, $N_{j,n\ell}$ and $\zeta_{j,n\ell}$ denote the normal functions of mass transport in x- and y- direction and displacement (downwards) of the upper face, respectively; the suffices 1 and 2 for j indicate respectively the upper and lower layer, the suffices n and m (1 to 6) and ℓ and p (all positive integers) represent respectively the mode and the order of the normal mode functions and the asterisk denotes the conjugate complex. In this equation, the first and the second three terms in the integral sign represent the sum of kinetic and potential energy of the flow in the upper and lower layer, respectively and the last term represent the effect of interaction of the upper and lower layer.

The quantities $C^{n\ell}(k')$ can be computed by substituting the normal mode functions for $n = m$ and $\ell = p$ into the left hand side of equation (44). However, the values of $C^{n\ell}(k')$ can be estimated without resorting to a complicated calculation. As discussed in

Sections 3 and 4, the normal mode functions are almost completely separated into barotropic and baroclinic mode in a range of k' much less than uninity or in a range of wave lengths from several to several thousands kilometers. Therefore, for the barotropic mode, the normal functions of the lower layer M_2, N_2 and ζ_2 can be expressed approximately by

$$M_2 \sim \frac{h_2}{h_1} M_1, \quad N_2 \sim \frac{h_2}{h_1} N_1 \quad \text{and} \quad \zeta_2 \sim \frac{\gamma - 1}{\gamma} \zeta_1 . \tag{45a}$$

In this case the mass transports in the whole depth become

$$M = M_1 + M_2 = \frac{h_1 + h_2}{h_1} M_1, \quad N = N_1 + N_2 = \frac{h_1 + h_2}{h_1} N_1 \tag{45b}$$

Substitution of (45a) into the left hand side of (44) leads to:

$$\begin{aligned} C^{n\ell} &\approx \int_0^\infty \left[\frac{h_1 + h_2}{\rho^2 h_1^2} \left(|M_{1,n\ell}|^2 + |N_{1,n\ell}|^2 \right) + |\zeta_{1,n\ell}|^2 \right] dx \\ &= \int_0^\infty \left[\frac{1}{\rho^2 g (h_1 + h_2)} \left(|M_{n\ell}|^2 + |N_{n\ell}|^2 \right) + |\zeta_{1,n\ell}|^2 \right] dx . \end{aligned} \tag{46}$$

The integral equals to the norm of the set of eigenfunctions (or normal mode functions) for a homogeneous water with the depth of $h_1 + h_2$ and is determined by Reid (1958) as his equation (64). (He used a notation $N_{\ell n}(k)$ instead of $C^{n\ell}$)

The normal functions of the lower layer for baroclinic mode are also given approximately by:

$$M_2 \sim \delta M_1 \quad N_2 \sim \delta N_1 \quad \text{and} \quad \zeta_2 \sim -\frac{1}{\delta} \zeta_1 \tag{47}$$

Substitution of (47) into the left hand side of (44) yields:

$$C^{n0} \approx \int_0^\infty \left[\frac{1}{\rho^2 g h_1} \left\{ |M_{1,n0}|^2 + |N_{1,n0}|^2 \right\} + |\zeta_{1,n0}|^2 + \frac{1}{\delta} |\zeta_{1,n0}|^2 \right] dx \tag{48}$$

The integral of the first three terms of the right hand side is the same as the norm of the normal functions for the upper layer only as treated by Reid (1958). The integral of the last term equals to $(2k\delta)^{-1}$.

As in case of a homogeneous water, the quantity :

$$P_{\ell,n}(k) = |2k\, C^{n\ell}(k)|^{-1} \tag{49}$$

indicates the weight function of partitioning of the energy spectrum among the six discrete modes (ℓ = 1 to 6). This function also satisfies the condition that:

$$\sum_{\ell=1}^{6} P_{\ell n}(k) = 1 \tag{50}$$

The weight functions for baroclinic mode are almost the same as those determined by Reid (1958) for a homogeneous water with the depth h_1. The approximate formula (48) shows that $C^{n\ell} \geq (2k\delta)^{-1}$. Therefore, for the baroclinic mode, we have

$$P_{n\ell} \leq \delta \tag{51}$$

This relation indicates that energy spectrum of an initial surface displacement or of forcing functions is distributed for the most part among three barotropic modes, because the weight functions for baroclinic mode are less than δ . Therefore, there is no favorable condition for exciting internal waves in a range of wave lengths of several thousands kilometers.

7. Concluding remarks

Mathematical treatment of edge waves in a two-layered rotating sea reveals to us two main features which may be important to the application of the theory. One is that no instability occurs to this system in a range of wave lengths of several to thousands kilometers. In a non-rotating system, shearing motion in a vertically stratified layer becomes unstable in a certain range of wave lengths as in Helmholz waves. Even in a rotating system, a mathematical model of cyclone waves and a simplified theory of meander of a wide ocean current predict that instability occurs in a two-layered system of geostrophic currents. Therefore, in this model there is no favorable range of wave lengths for which some disturbances may excite large internal waves.

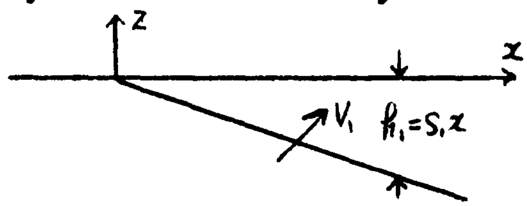
Another feature is that energy of an initially static mound of water and of disturbing forces due to atmospheric pressure or wind stresses is distributed for the most part among three barotropic modes. This again indicates that internal waves are not favorably excited by such disturbances with an order of dimension corresponding to a range of wave lengths considered. This is in accordance with a general theory developed by Veronis and Stommel (1958), who concluded that the ocean responds to variable winds principally as a homogeneous body of water and for periods shorter than several weeks. Therefore, possibility of favorable excitation of internal waves in this system may be found outside a range of wave lengths discussed here. In a range of shorter wave lengths, however, the effect of Coriolis' force is negligible. In a range of longer wave lengths, the effect of change of Coriolis' coefficient may become important. In either range the present model is not adequate.

Acknowledgement: Typing of the manuscript by Mrs. Suzy Bowman is greatly appreciated.

Appendix A

Stommel (1953, 1958) discussed a perturbation problem for a wide, geostrophic current V_1 in the upper layer of two-layered ocean. He obtained a result that there is a narrow range of wave numbers about $k = f/\sqrt{2}\,V_1$ in which waves are unstable, in contrary to the present work. Although he assumed that the perturbation is uniform in x - direction (cross current direction), his model may be considered as a modified use of the present theory.

It is assumed that the uniform geostrophic current flows in the wedgeshaped upper layer, below which the motionless lower layer extends infinitely downwards. (See the attached figure) The coordinate system is taken as the same as in the text, but no coastal boundary is needed in this problem Equations of motion and continuity for the upper layer are the same as (1) and (2) of the text. The assumption of no motion in the lower layer leads to:



$$\zeta_2 = -\delta^{-1}\zeta_1 \;. \qquad \text{(A1)}$$

Substituting (4) and (5) into (2) and using (A1), we have :

$$\frac{d}{dx}\left(h\frac{d\zeta}{dx}\right) - \left\{\frac{fkS}{\sigma_1} + k^2 h + \frac{1+\delta}{g\delta}\Delta_1\right\}\zeta = 0 \qquad \text{(A2)}$$

in which $\sigma_1 = \sigma - kV$, $\Delta_1 = f^2 - \sigma_1^2$ and the suffix 1 for S , h , ζ and V is omitted.

Assuming that ζ is independent on x and that h in the coefficient of ζ in equation (A2) is constant, Stommel (1954, 1958) derived a cubic equation of σ by equating the coefficient of ζ to zero. However, if h is taken as Sx throughout the whole space, equation (A2) can be written as:

$$\frac{d}{dx}\left(x\frac{d\zeta}{dx}\right) - \left\{\frac{k}{\omega_1} + \frac{f(1-\omega_1^2)}{V} + k^2 x\right\}\zeta = 0 \qquad \text{(A3)}$$

in which $\omega_1 = (\sigma - kV)/f$ and $V = g\delta S/f$ is used.

The normal mode of ζ which is finite at x = 0 and ∞ corresponds to ω_1 satisfying

$$k\omega_1^{-1} + f(1-\omega_1^2)\nabla^{-1} = -(2n+1)|k| \tag{A4}$$

This equation can be transformed into

$$\omega_1^3 - \{(2n+1)k' + 1\}\omega_1 - k' = 0 \tag{A5}$$

where $k' = kV/f^{-1}$ as defined in the text.

By taking $\omega_1 = -(k'/2)^{1/3}W$, (A5) is changed into:

$$W^3 + 2 = PW \tag{A6}$$

where $P = (2/k')^{-2/3}\{(2n+1)k'+1\}$ (A7)

when $P < 3$, there are complex roots of (A6). However, as easily seen, $P \geqslant 3$ for all positive k'. (The equality occurs for k'=2 and n = 0). Therefore, in the present model there is no range of k' in which waves are unstable.

When there is a geostrophic current in the lower layer as in the model of the text, k' in equation (A5) should be changed into $(1+\varepsilon)k'$, in which $\varepsilon = -V_2/V_1$. Since $(1+\varepsilon)k'$ is always positive as discussed in the text, the same reasoning as above is valid and the waves are always stable.

It is noted that equation (A5) is reduced to the frequency equation (30) of Reid's paper (1958), when ω and k' are replaced by $-\omega/f$ and $\delta g S k f^{-1}$, respectively. Therefore, mathematical features of the roots of (A5) are the same as those of the barotropic mode discussed by him. Discussions on phase velocity and group velocity for each of the three barotropic modes are also applicable to the baroclinic mode, when the velocities are interpreted as those referred to the coordinates moving with the flow in the upper layer and the wave number k for the barotropic mode is replaced by $\delta^{-1}(1+\varepsilon)^{-1}k$.

Appendix B

Equations (1) and (2) yields the equations of motion and continuity for a mode ℓ and order n such as

$$-i\sigma_{j,n\ell} M_{j,n\ell} - f N_{j,n\ell} = g h_j \frac{d\eta_{j,n\ell}}{dx} , \quad j=1, 2 , \tag{B1}$$

$$f M_{j,n\ell} - i\sigma_{n\ell} N_{j,n\ell} = i k g h_j \eta_{j,n\ell}, \; j=1, 2 , \tag{B2}$$

$$\frac{d M_{j,n\ell}}{dx} + i k N_{j,n\ell} = i\sigma_{j,n\ell}\phi_{j,n\ell} , \; j=1, 2 . \tag{B3}$$

If $\underline{n}$ and $\underline{l}$ are replaced by $\underline{m}$ and $\underline{p}$, respectively and all quantities are changed into their complex conjugate, the following equations are obtained:

$$i\sigma_{j,mp} M^*_{j,mp} - f N^*_{j,mp} = g h_j \frac{d\eta^*_{j,mp}}{dx} , \quad j=1, 2 \tag{B4}$$

$$f M^*_{j,mp} + i\sigma_{j,mp} N^*_{j,mp} = -i k g h_j \eta^*_{j,mp} \quad j=1, 2 \tag{B5}$$

$$\frac{d M_{j,mp}}{dx} - i k N^*_{j,mp} = -i\sigma_{j,mp}\phi^*_{j,mp} \quad j=1, 2 \tag{B6}$$

where the asterisk denotes the complex conjugate.

If equations (B1), (B2), (B3), (B4), (B5) and (B6) are multiplied respectively by $M^*_{j,mp}/\rho_j\rho_2 g h_j$, $N^*_{j,mp}/\rho_j\rho_2 g h_j$, $\frac{\rho_j}{\rho_2}\zeta^*_j$, $M_{j,n\ell}/\rho_j\rho_2 g h_j$

$N_{j,n\ell}/\rho_j\rho_2 g h_j$ and $\rho_j/\rho_2\,\zeta_j$ and all the resultants are summed up both for j = 1 and 2, the following equation is obtained

$$\begin{aligned} i(\sigma_{mp}-\sigma_{n\ell})\Big[&\frac{1}{\rho_1\rho_2 g h_1}\{M_{1,n\ell}M^*_{1,mp}+N_{1,n\ell}N^*_{1,mp}\} + \frac{\rho_1}{\rho_2}\zeta_{1,n\ell}\zeta_{1,mp} + \frac{1}{\rho_2 g h_2}\{M_{2,n\ell}M^*_{2,mp} + \\ &+ N_{2,n\ell}N^*_{2,mp}\} + \frac{\Delta\rho}{\rho_2}\zeta_{2,n\ell}\zeta_{2,mp} + \frac{\rho_1}{\rho_2}(V_1-V_2)k\zeta_s g\frac{(2n+1)\sigma_{1,n\ell}\sigma_{1,mp}(\sigma_{1,mp}+\sigma_{1,n\ell})+f^3}{\sigma_{1,n\ell}\sigma_{1,mp}(f^2-\sigma^2_{1,n\ell})(f^2-\sigma^2_{1,mp})}\zeta_{1,mp}\zeta_{1,n\ell} - \\ &-\frac{\rho_1}{\rho_2}(V_1-V_2)k^3\zeta_s g\frac{f(\sigma^2_{1,mp}+\sigma_{1,mp}\sigma_{1,n\ell}+\sigma^2_{1,n\ell})}{\sigma_{1,n\ell}\sigma_{1,mp}(f^2-\sigma^2_{1,n\ell})(f^2-\sigma^2_{1,mp})}\Big] = \frac{1}{\rho_2}\frac{d}{dx}\{\zeta^*_{1,mp}M_{1,n\ell}+\zeta_{1,n\ell}M^*_{1,mp}+\eta^*_{2,mp}M_{2,n\ell}+\eta_{2,n\ell}M^*_{2,mp}\} \end{aligned} \tag{B7}$$

in which equation (14) is used.

When this equation is integrated from 0 to ∞ with respect to x, the right hand side of the relation vanishes because of boundary conditions at x = 0 and at x = ∞ . Therefore, the expression in brackets of the left hand side of (B7) must vanish except for m = n and p = 1.

$$\frac{1}{\rho_1\rho_2 g k_1}\{M_{1,n\ell}M^*_{1,mp} + N_{1,n\ell}N^*_{1,mp}\} + \frac{\rho_1}{\rho_2}\zeta_{1,n\ell}\zeta_{1,mp} + \frac{1}{\rho_2^2 g k_2}\{M_{2,n\ell}M^*_{2,mp} + N_{2,n\ell}N^*_{2,mp}\} + \frac{\Delta\rho}{\rho_2}\zeta_{2,n\ell}\zeta_{2,mp} +$$
$$+ \frac{\rho_1}{\rho_2}(V_1 - V_2)k^2 S_1 g \frac{(2n+1)\sigma_{1,n\ell}\sigma_{1,mp}(\sigma_{1,mp}+\sigma_{1,n\ell}) - f(\sigma_{1,mp}^2 + \sigma_{1,mp}\sigma_{1,n\ell} + \sigma_{1,n\ell}^2) + f^3}{\sigma_{1,n\ell}\sigma_{1,mp}(f^2-\sigma_{1,n\ell}^2)(f^2-\sigma_{1,mp}^2)}\zeta_{1,mp}\zeta_{1,n\ell} \qquad \text{(B8)}$$
$$= \begin{cases} 0 & n \neq m \text{ or/and } \ell \neq p \\ C^{n\ell}(k) & n \neq m \text{ and } \ell = p. \end{cases}$$

The quantity $C^{n\ell}(k)$ can be calculated by using the following relations and equations (4), (5) and (14).

$$(n+1)\zeta_{1,n+1} - (2n+1-2kx)\zeta_{1,n} + n\zeta_{1,n-1} = 0 \qquad \text{(B9)}$$
$$\zeta'_{1,n} - \zeta'_{1,n-1} + k(\zeta_{n-1} + \zeta_n) = 0$$
$$x\zeta'_{1,n} = n(\zeta_{1,n} - \zeta_{1,n-1}) - kx\zeta_{1,n}$$

$$\zeta_{1,n} = \frac{1}{n!}e^{-kx}L_n(2kx) \qquad \text{(B10)}$$

$$\int_0^\infty \zeta_{1,n}\zeta_{1,m}\,dx = \begin{cases} 1/2\,k & m = n \\ 0 & m \neq n \end{cases}$$

The result is:

$$C^{n\ell} = \frac{\rho_1/\rho_2}{2k(f^2-\sigma_{1,n\ell}^2)^2}\{(2n+1)S_1 g k(\sigma_{1,n\ell}^2 + f^2) + 2fS_1 g k\sigma_{1,n\ell}\} + \frac{\rho_1/\rho_2}{2k} +$$
$$+ \frac{1}{2k(f^2-\sigma_{2,n\ell}^2)}\left[\frac{\{(2n+1)\sigma_{1,n\ell}+f\}\delta S_1 g k}{\sigma_{1,n\ell}(f^2-\sigma_{1,n\ell}^2)} + 1\right]^2\{(2n+1)S_2 g k(\sigma_{2,n\ell}^2+f^2) + 2fS_2 g k\sigma_{2,n\ell}\} + \qquad \text{(B11)}$$
$$+ \frac{\Delta\rho/\rho_2}{2k}\left[\frac{\{(2n+1)\sigma_{1,n\ell}+f\}\delta_1 g k}{\sigma_{1,n\ell}(f^2-\sigma_{1,n\ell}^2)} + 1\right]^2 + \frac{\rho_1/\rho_2(V_1-V_2)k^2 S_1 g\{2(2n+1)\sigma_{1,n\ell}^3 - 3f\sigma_{1,n\ell}^2 + f^3\}}{\sigma_{1,n\ell}^2(f^2-\sigma_{1,n\ell}^2)^2}.$$

References

Eckart, Carl.
1951. Surface waves on water of variable depth. SIO Wave Report No. 100 Ref. 51-12, 99 pp.

1960. Hydrodynamics of Oceans and Atmosphere. Pergamon Press, 290 pp.

Godske, C. L., T. Bergeron, J. Bjerknes, R. C. Bundgaard.
1957. Dynamic meteorology and weather forecasting, American Meteorological Society, 800 pp.

Greenspane, H. P.
1956. The generation of Edge waves by moving pressure distribution. J. Fluid Mech., 1: 574-592.

Ichiye, Takashi.
1963. Internal waves over a continental shelf, Florida State University Tech. Rep. No. 3, 21 pp.

Kajura, Kinjirō.
1958. Effect of Free and Forced Waves, J. Mar. Res., 16:145-157.

Kelvin, Lord.
1879. On gravitational oscillations of rotating water, Proc. Roy. Soc. Edinb. 10, 92.

Reid, R. O.
1958. Effect of Coriolis force on edge-waves (I) Investigation of the normal modes. J. Mar. Res., 16:109-144.

Stokes, G. G.
1946. Mathematical and physical papers, I, Cambridge Univer. Press pp. 167-168.

Stommel, Henry.
1953. Examples of the possible role of inertia and stratification in the dynamics of the Gulf Stream System, J. Mar. Res., 12:184-195.

1958. The Gulf Stream - a physical and dynamical description, University of California Press, 202 pp.

Ursell, F.
1952. Edge waves on a sloping beach, Proc. Roy. Soc., (A) 214:79-97.

G. Veronis and H. Stommel.
1956. The action of variable wind stresses on a stratified ocean, J. Mar. Res. 15:43-75.